Math 102

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Goals Today - Qualitative analysis of Autonomous Differential Equations

- What is an autonomous DE?
- Example: the logistic equation
- Solutions to an autonomous DE
- Practice...

Autonomous Differential Equations

Autonomous	Not Autonomous
$\frac{dP}{dt} = 3P$	$\frac{dP}{dt} = t$
$\frac{dP}{dt} = 20P - P^2$	$\frac{dP}{dt} = t^2P + 2P$
$\frac{dP}{dt} = \frac{P^3-3P}{2P^2+1}$	$\frac{dP}{dt} = \frac{t^2+t}{P}$
$\frac{dP}{dt} = \sin(P)$	$\frac{dP}{dt} = e^{3t}$

An autonomous differential equation has the form

$$\frac{dP}{dt} = f(P)$$

Such differential equations are also called time-independent.

Logistic Growth

$$\frac{dP}{dt} = aP - bP^2$$

where a and b are positive constants.

This equation models population growth, i.e. if $P(t)$ is the size of a population at time t.

The equation above has two steady states: 0 and $\frac{a}{b}$. $\frac{a}{b}$ is called the carrying capacity.

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We'll pick specific constants, and analyze $\frac{dP}{dt} = P^2 - 20P$.

$$\frac{dP}{dt} = 20P - P^2$$

Question: What are the steady states for this differential equation?

- $P(t) = 0$ and $P(t) = 20t$.
- $t = 0$ and $t = 20$.
- $P(t) = 0$ and $P(t) = 20$.
- $P(t) = e^{0t}$ and $P(t) = e^{20t}$.

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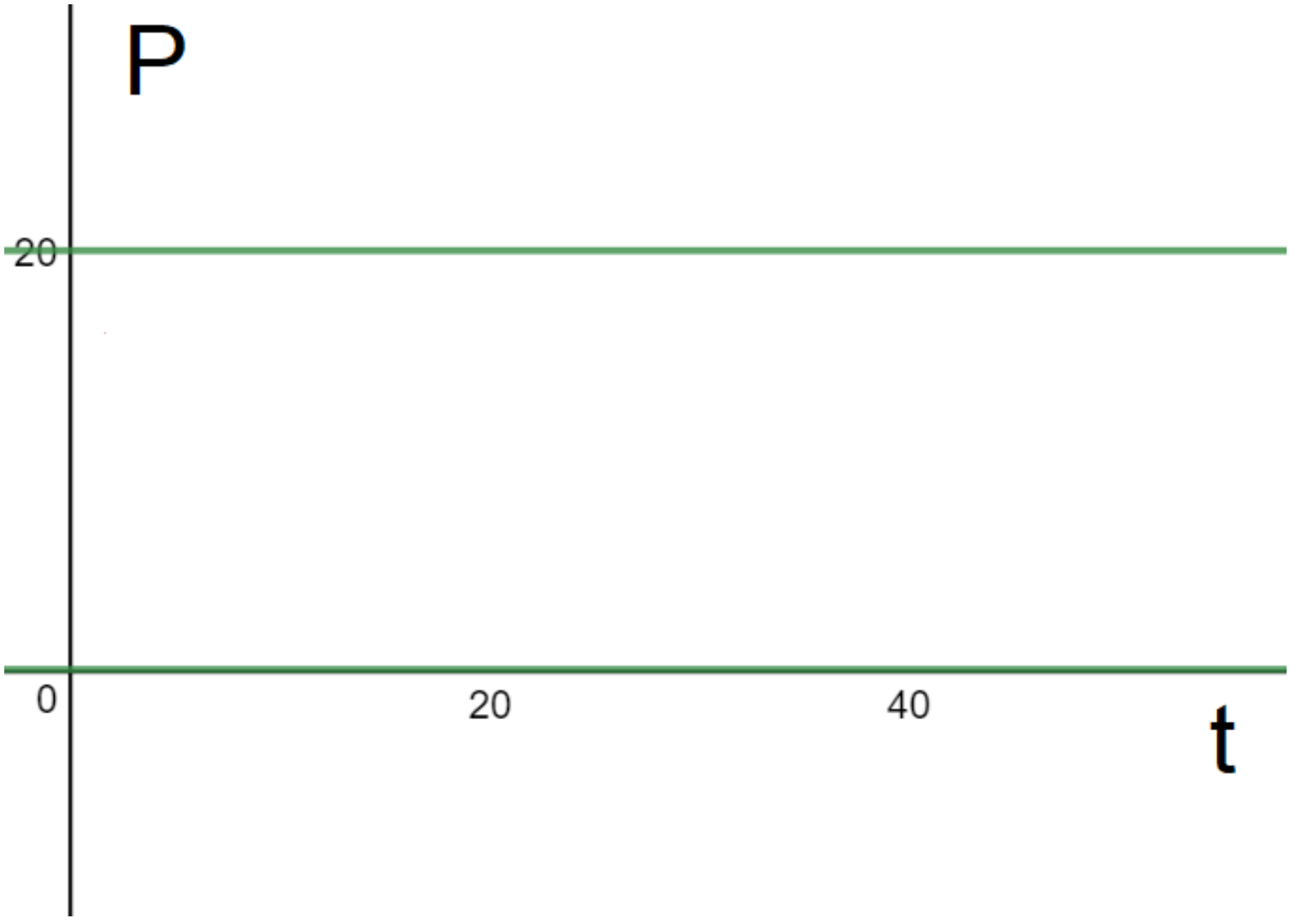

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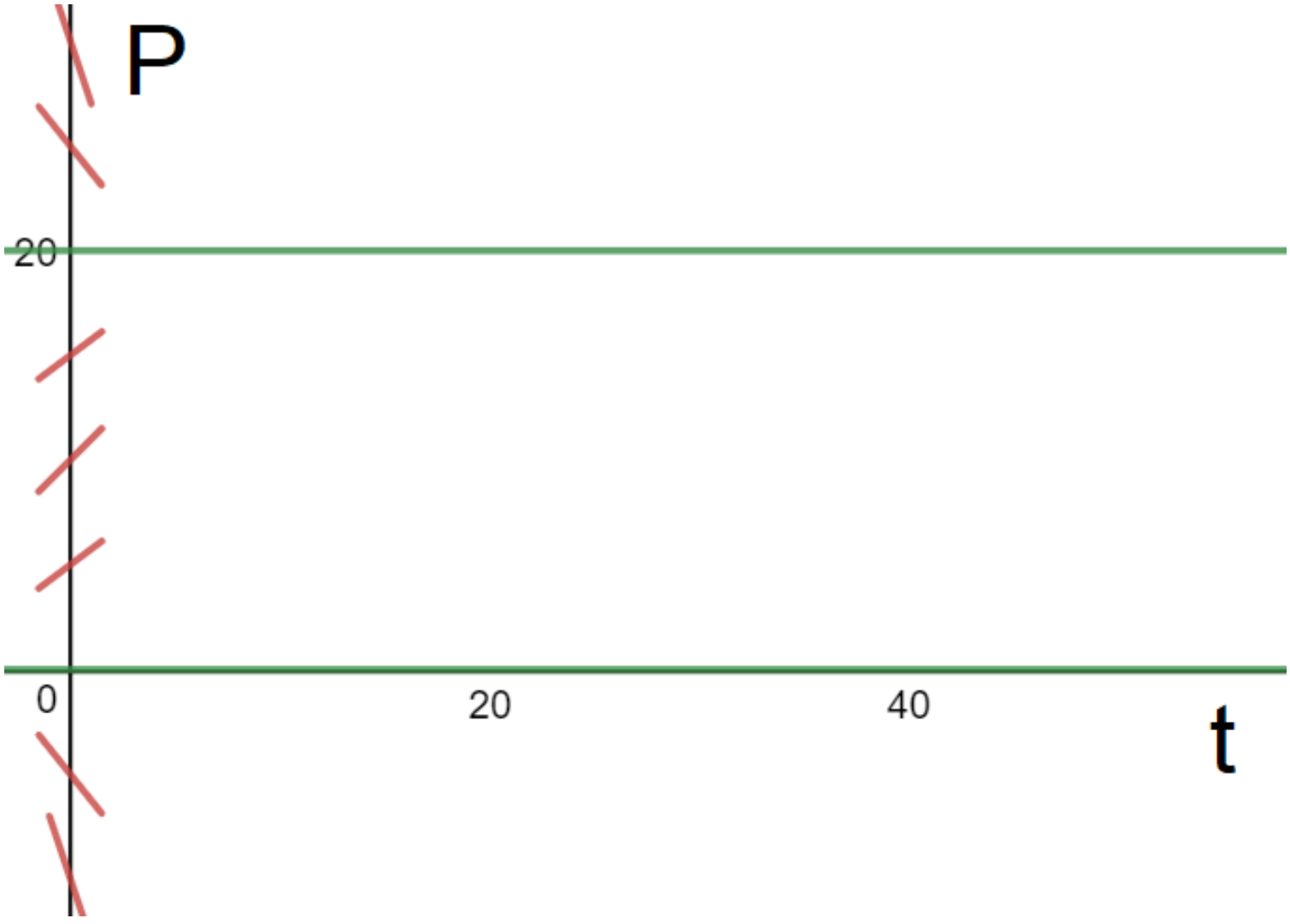
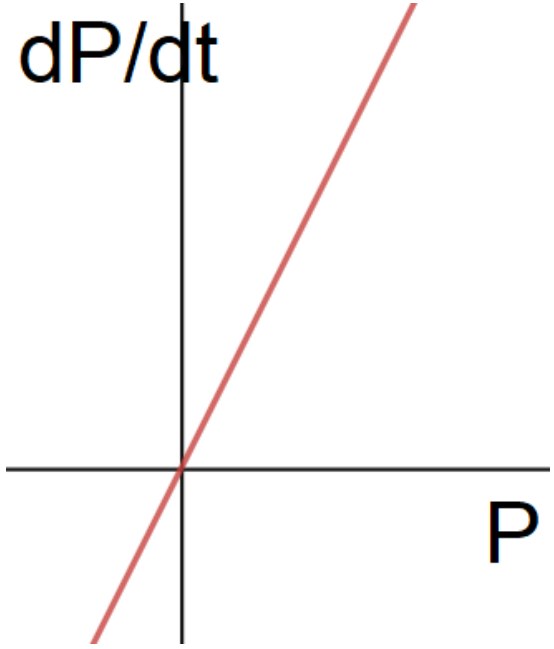
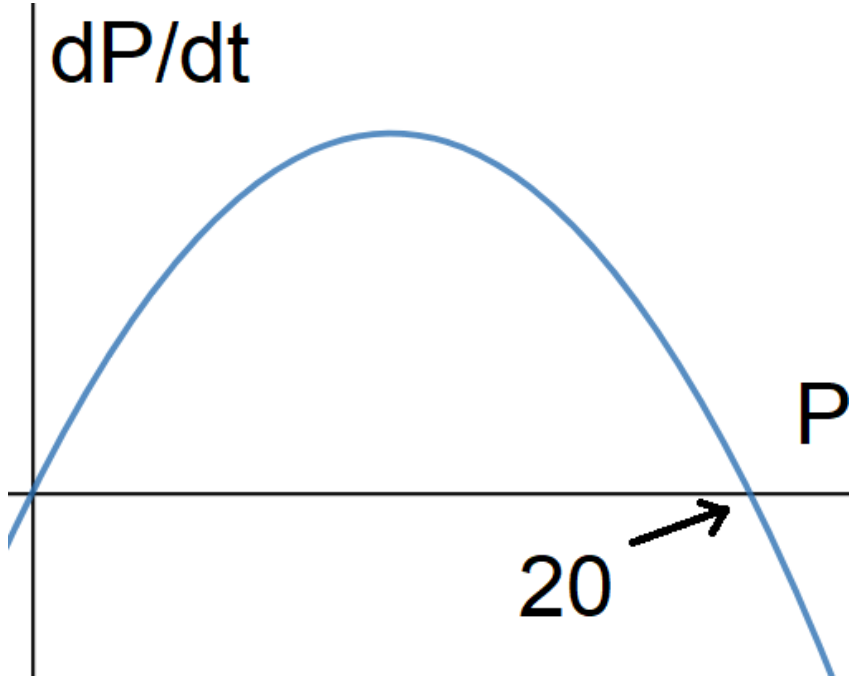


State Space

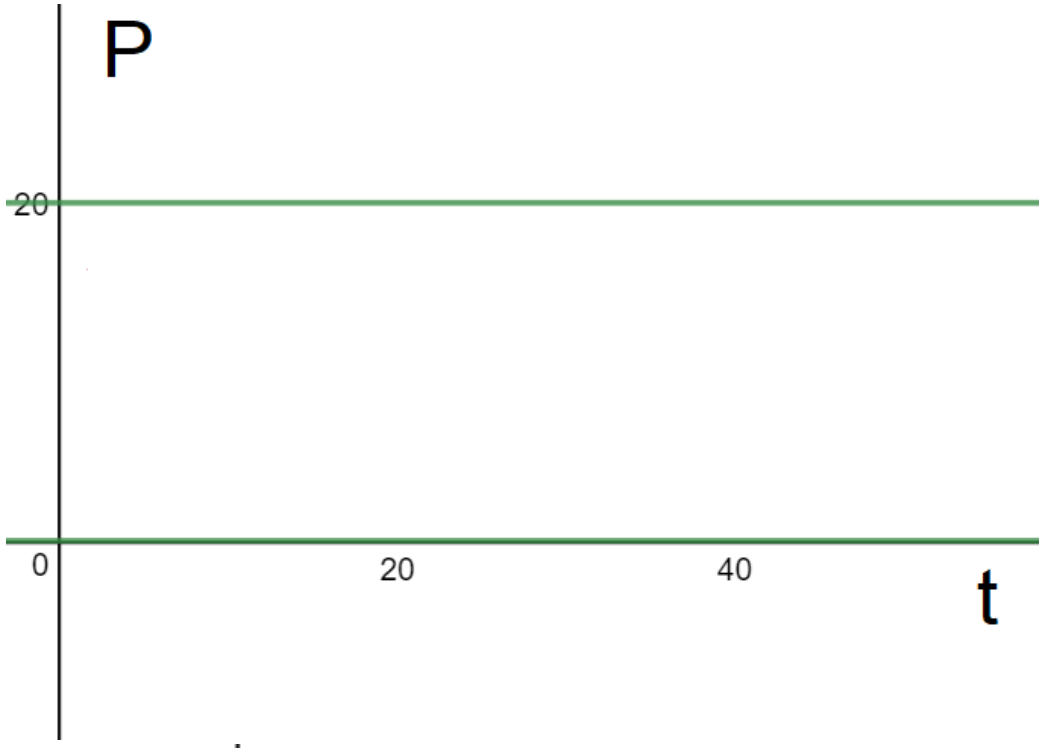
For an autonomous differential equation, the state space is a graph of $\frac{dP}{dt}$ against P.

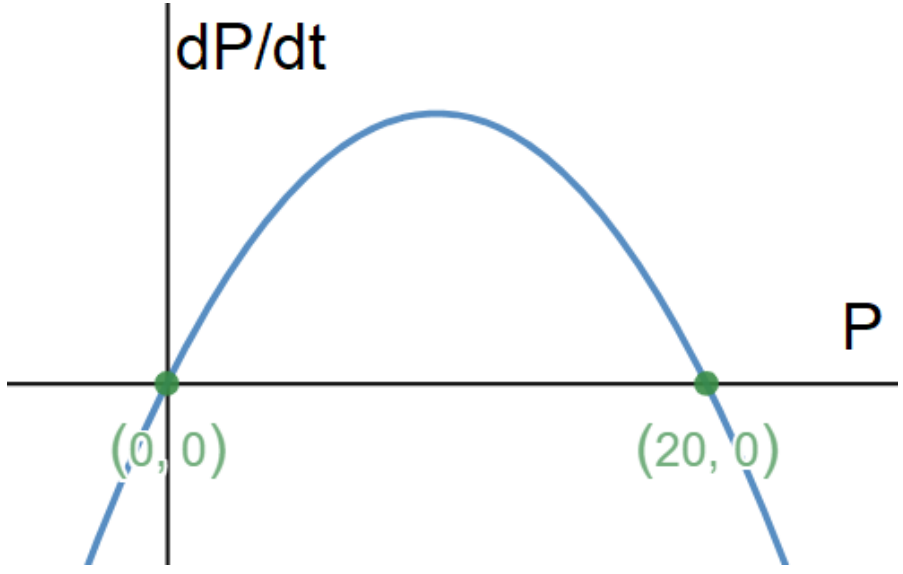


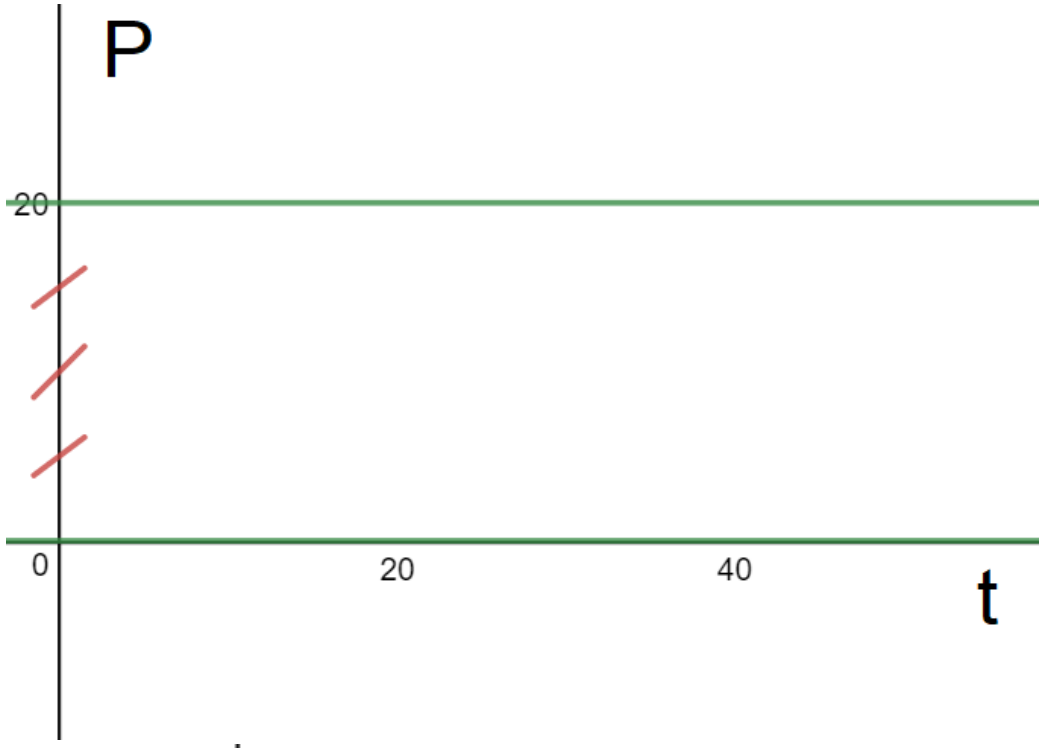
$$\frac{dP}{dt} = 5P$$

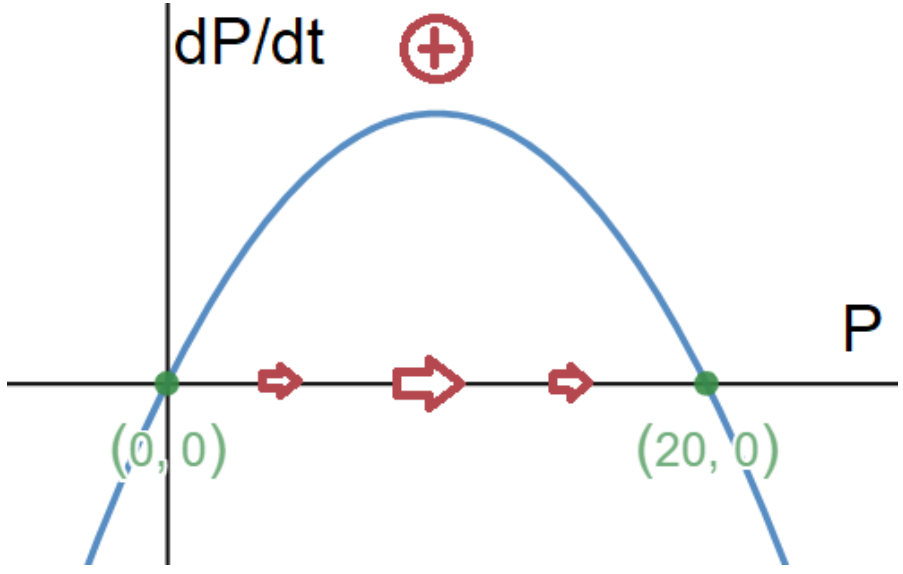


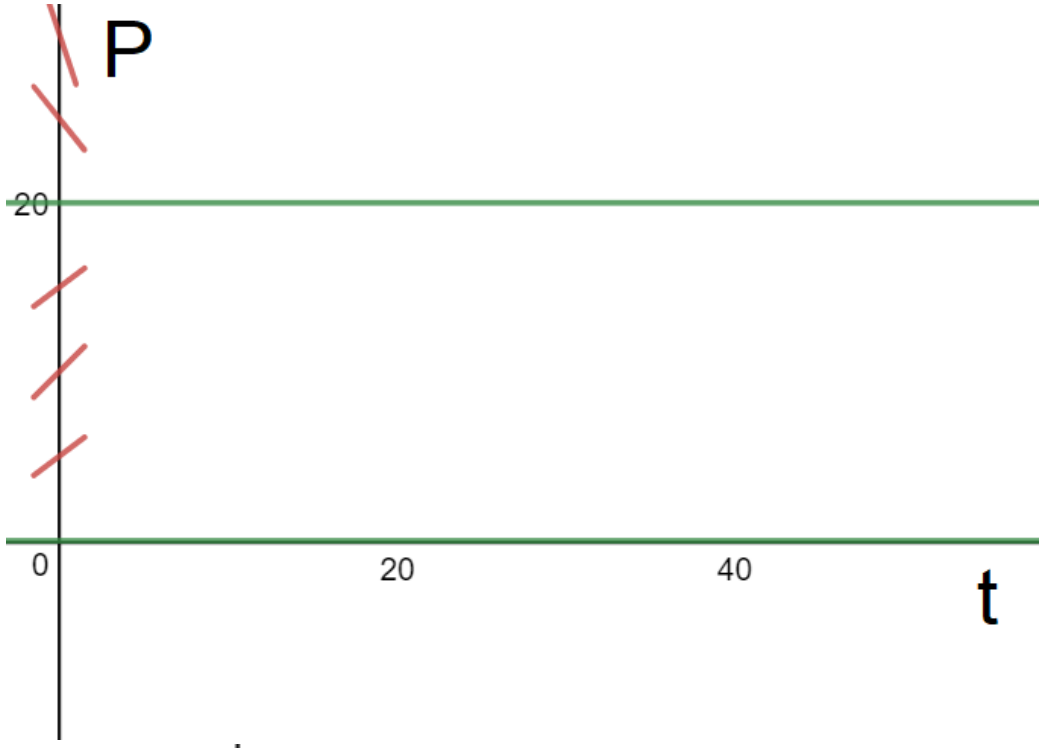
$$\frac{dP}{dt} = 20P - P^2$$

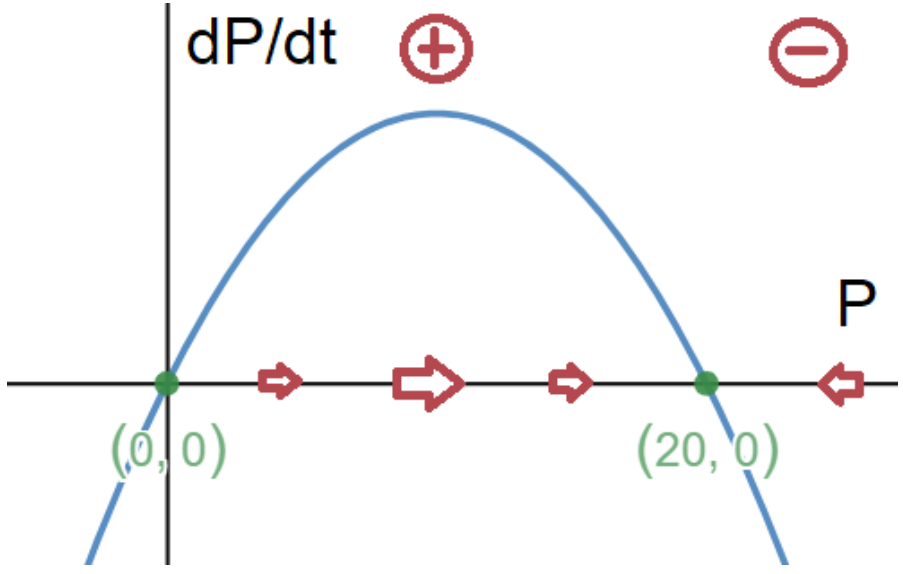

P
20
0
20
40
t

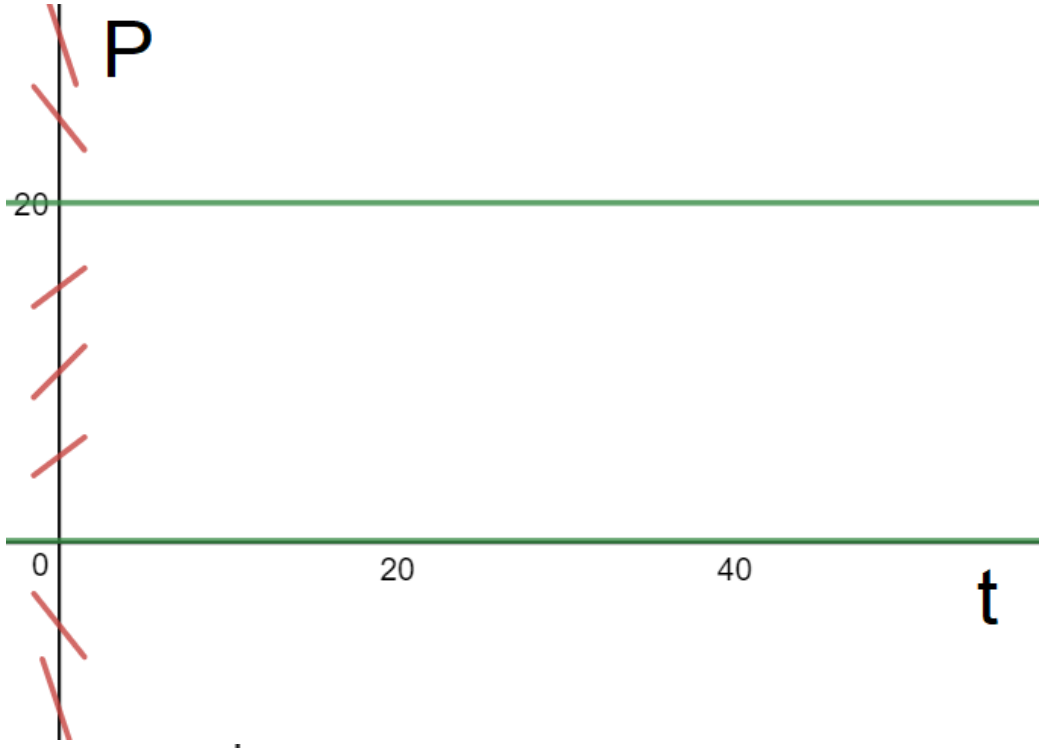

dP/dt
P
(0, 0)
(20, 0)

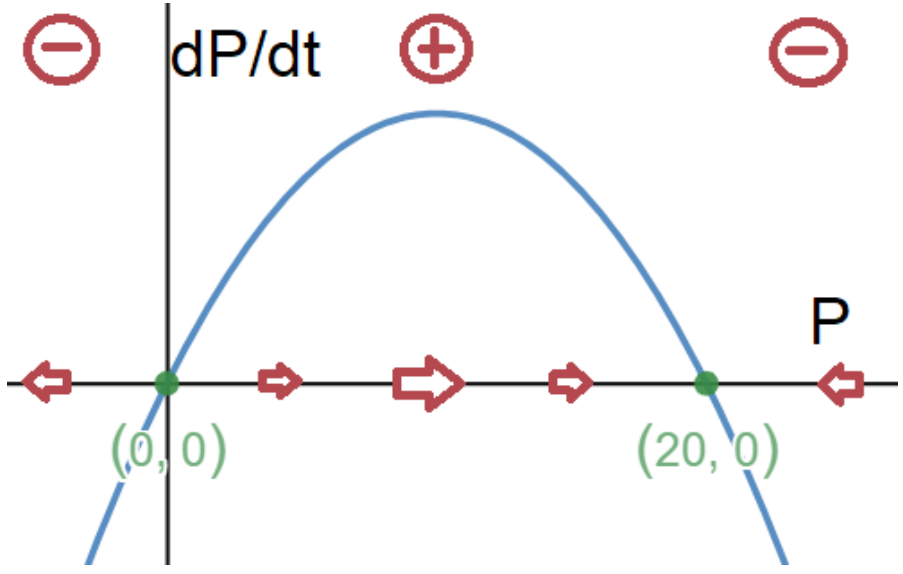

P
20
0
20
40
t


dP/dt
P
(0, 0)
(20, 0)

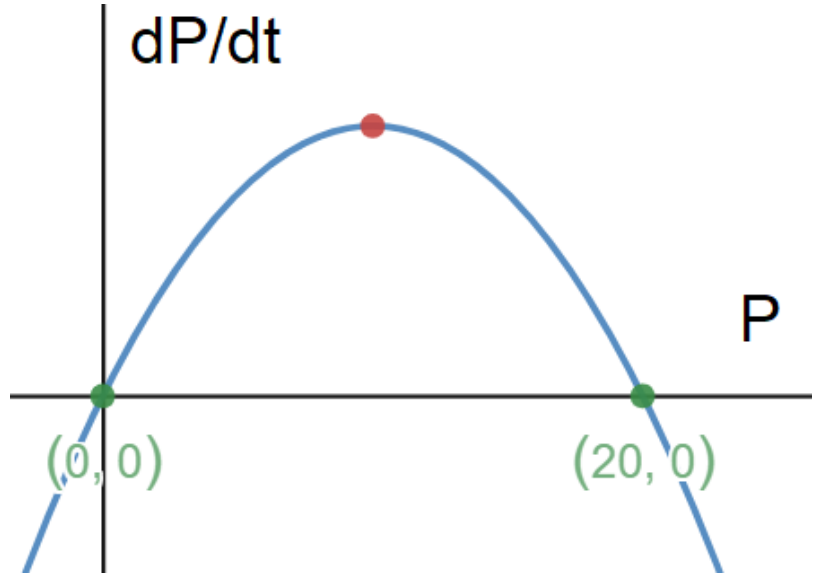

P
20
0
20
40
t


dP/dt
+
−
P
(0, 0)
(20, 0)


P
20
0
20
40
t


dP/dt
P
(0, 0)
(20, 0)

Question: What does the red point on the graph correspond to?



- When $P = 10$, the population doesn't change.
- When $P = 10$, the population is growing at the maximum possible rate.
- When $t = 10$, the population is growing at the maximum possible rate.
- Confused, help!

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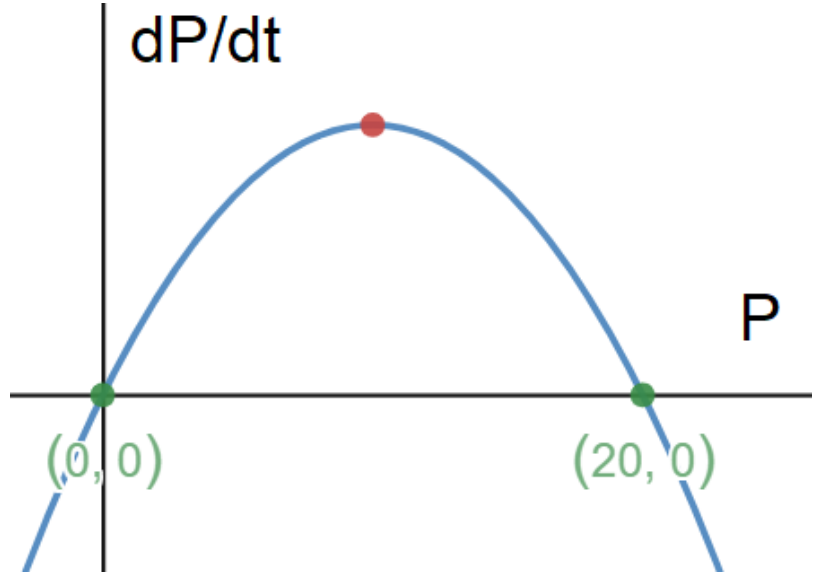

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Sketching solutions

$$\frac{dP}{dt} = 20P - P^2$$

Question: True or false: there can be two solutions which cross each other.

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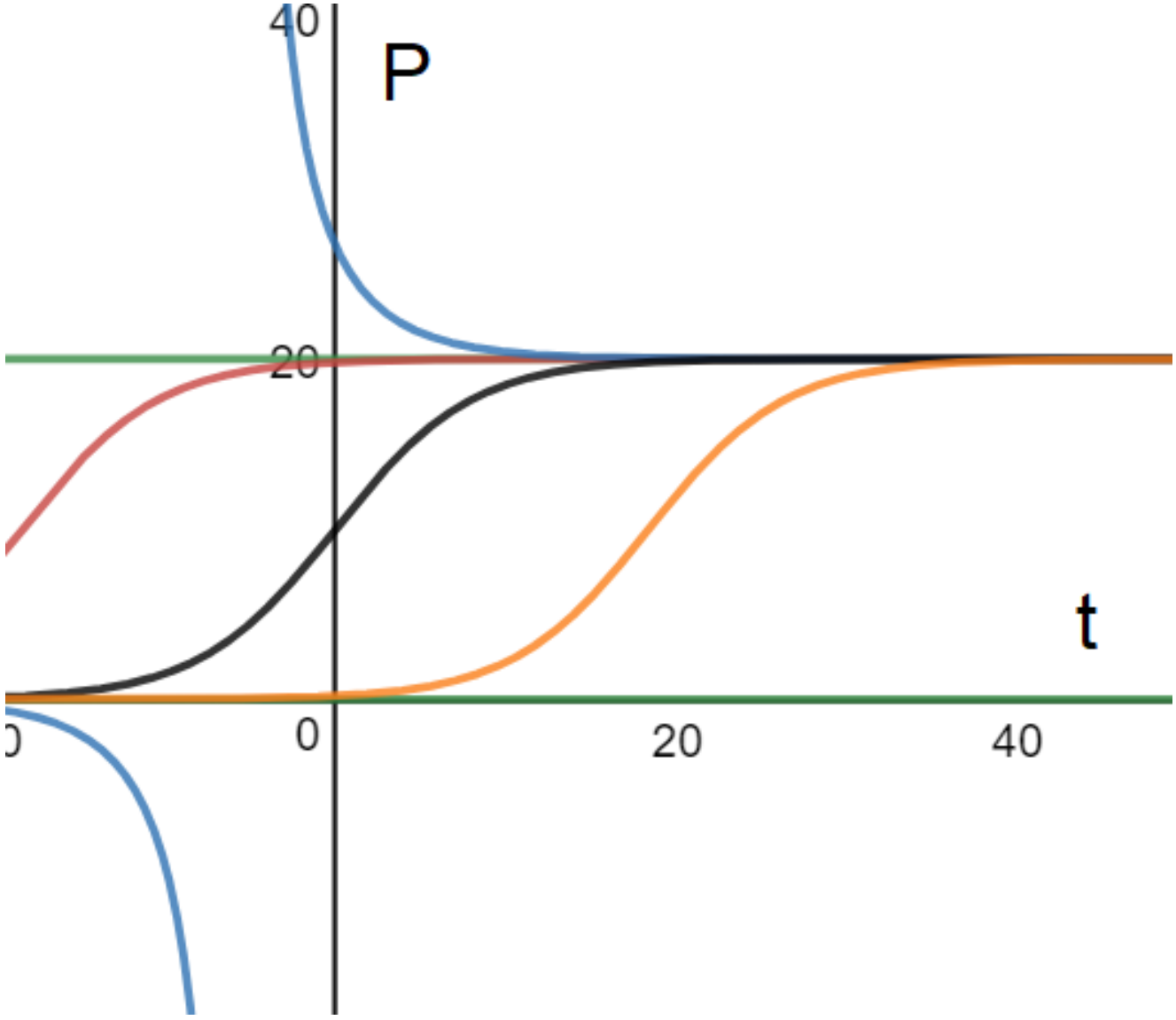
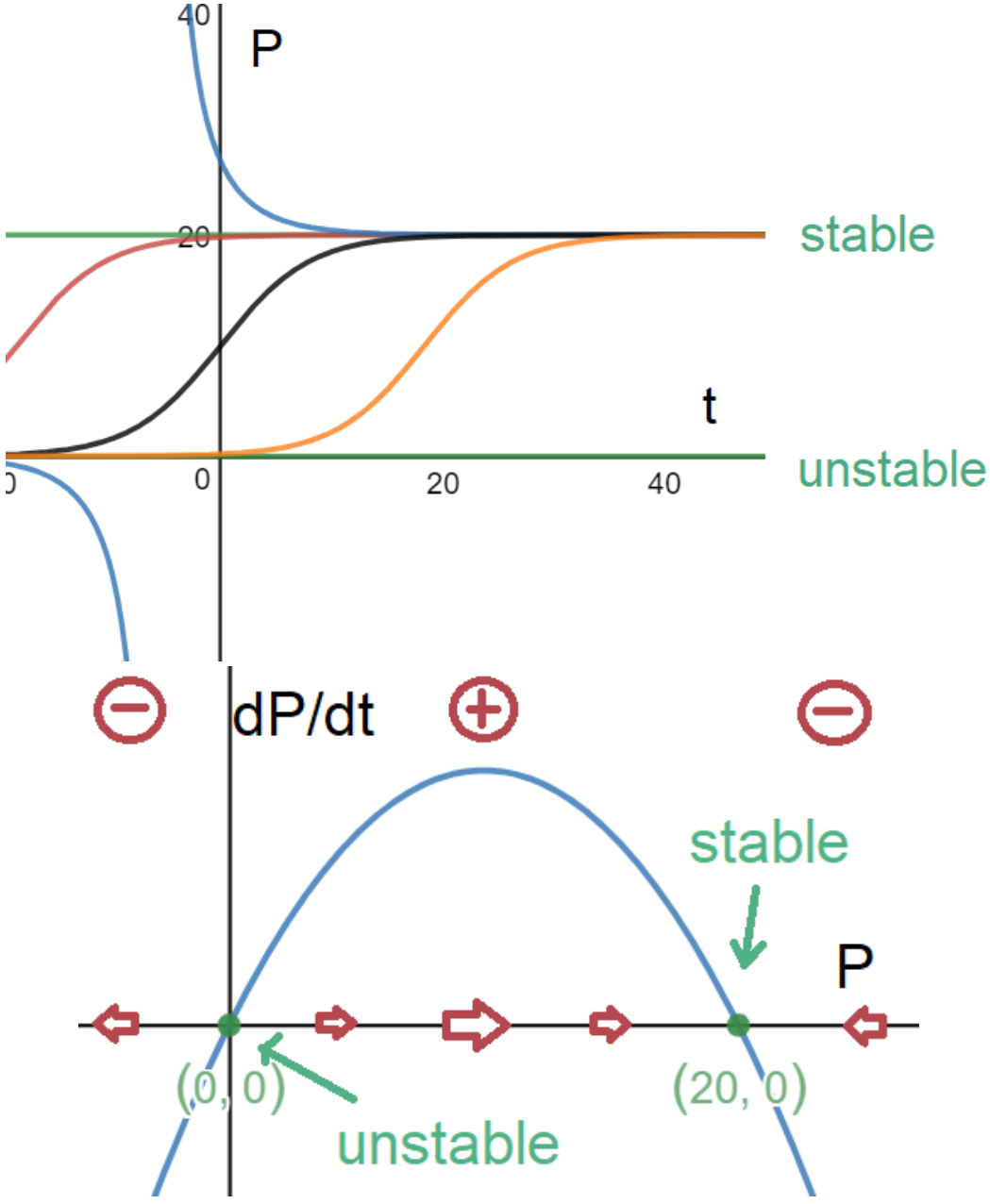
Question: True or false: there can be two solutions which cross each other.

False! If two solutions crossed each other, then at the crossing point there would be two possible values of $\frac{dP}{dt}$!

Sketching Solutions




40
P
20
stable
t
unstable
0
20
40
dP/dt
stable
P
(0, 0)
(20, 0)
unstable

Qualitative Analysis of Autonomous DE's

- Calculate the steady states. Draw these trajectories.
- Calculate whether $\frac{dy}{dt}$ is positive or negative in each region.
  - It maybe help to graph $\frac{dy}{dt}$ vs y, i.e the phase line.
  - Stable and unstable steady states.
- Sketch solutions.
  - No two solutions can cross.
  - If $y(t)$ is one solution, then $y(t+C)$ is a solution for any constant C. This means that any horizontal translate of a solution is also a solution.